The Stieltjes Convolution and a Functional Calculus for Non-negative Operators

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Abstract

In this paper we present an approach to the multidimensional distributional Stieltjes transform that allows us to define a convolution operation on our classes of Stieltjes-transformable distributions.

As an application, we develop a powerful and intuitive functional calculus for (possibly multiple) non-negative operators with which one can easily prove a variety of operator equations, even for non-commuting operators and on non-complex Banach spaces.

Two representation theorems that identify our classes of distributions as finite sums of derivatives of functions that fulfill certain estimates are essential throughout this paper.

Key words: Stieltjes transform, distribution, convolution, non-negative operator, functional calculus

1 Motivation

One aspect of the classical Stieltjes transform

$$\mathcal{S}(f)(a) := \int_0^\infty f(t)\,\frac{1}{t+a}\,dt \qquad (a>0) \tag{1}$$

is that a satisfactory notion of a convolution operation, i.e. of an operation $*$ with $\mathcal{S}(f*g)=\mathcal{S}(f)\cdot\mathcal{S}(g)$, cannot be defined: In [1] Hövel and Westphal showed that there exist Stieltjes-transformable functions f and g such that $\mathcal{S}(f)\cdot\mathcal{S}(g)$ is not the classical Stieltjes transform of any other function. An

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Preprint submitted to Elsevier Science *1 February 2006*

easier counterexample for measures instead of functions will be shown in Example 1. But they found a way of defining a convolution operation at least for functions f and g with certain additional properties that are oftentimes hard to check (or not fulfilled).

As we will see in this paper (and as shown previously by Th. Schwartz in [3]), it turns out that this obstacle can be overcome if one looks at the problem from a higher viewpoint, namely by generalizing the Stieltjes transform to distributions. This means that in the counterexample mentioned above the Stieltjes-convolution of f and g is in fact a Stieltjes-transformable distribution that is not a function.

In the last decades various approaches to extend the classical Stieltjes transform to certain distribution classes were made successfully. These classes are either defined as the topological dual of some test function space, or as the set of finite sums of derivatives of certain measures, i.e. as distributions of the form

$$S = \sum_{k=0}^{K} D^k \mu_k \tag{2}$$

for certain measures μ_k. A survey can be found in the book by Pilipović [2, chapter 4].

Since the second approach is inadequate for defining the convolution operation, Th. Schwartz chose the first (and less intuitive) one for his definition of the distributional Stieltjes transform. This however leads to serious problems when one actually wants to work with this newly developed tool:

For our goal to develop a functional calculus for non-negative operators we need the distributions to have a representation of the form (2) for measures μ_k on $(0, \infty)$ with

$$\int_0^\infty \lambda^{-(k+1)} d|\mu_k|(\lambda) < \infty. \tag{3}$$

This would motivate us to map the Stieltjes transform f of such a distribution,

$$f(a) = \mathcal{S}(S)(a) := S(\tfrac{1}{\cdot + a}) = \sum_{k=0}^{K} k! \int_0^\infty \Big(\frac{1}{\lambda + a}\Big)^{k+1} d\mu_k(\lambda) \qquad (a > 0),$$

to the operator

$$\mathcal{R}_A(f) := \sum_{k=0}^{K} k! \int_0^\infty \Big(\frac{1}{\lambda + A}\Big)^{k+1} d\mu_k(\lambda). \tag{4}$$

Here A is assumed to be a non-negative operator on a Banach space X, that means that A is closed, $(-\infty, 0) \subset \varrho(A)$ (the resolvent set of A) and $\sup_{\lambda>0} ||\lambda(\lambda + A)^{-1}|| < \infty$. The convolution comes into play when we want to prove the relation $\mathcal{R}_A(f \cdot g) = \mathcal{R}_A(f)\mathcal{R}_A(g)$ for $f = \mathcal{S}(S)$ and $g = \mathcal{S}(T)$ because then we need to make sure that $f \cdot g = \mathcal{S}(S) \cdot \mathcal{S}(T)$ is again the Stieltjes transform of a distribution, that is $\mathcal{S}(S * T)$.

In this paper we present an approach to the distributional Stieltjes transform that follows the two apporaches at the same time: Starting from the definition of the space $\mathcal{D}_{S^*}(\mathbb{R}_+)'$ of all strongly Stieltjes-transformable distributions as the topological dual of a certain test function space, we prove that $\mathcal{D}_{S^*}(\mathbb{R}_+)'$ is the space of all distributions of the form (2) with the estimates (3). While the definition becomes important for the proofs of some basic properties of the convolution, the representation theorem is essential for other proofs and especially later on for the definition of the functional calculus.

We will then weaken the estimates for the measures (3) and extend $\mathcal{D}_{S^*}(\mathbb{R}_+)'$ to the class $\mathcal{D}_S(\overline{\mathbb{R}}_+)'$ of all Stieltjes-transformable distributions so that our approach extends both the classical Stieltjes transform and the classical Stieltjes convolution presented in [1].

We will also treat the general n-dimensional case. By doing this, we will get a multidimensional functional calculus with which one can prove operator equations involving the resolvents of several non-negative operators A_i ($i = 1, \ldots, n$) at a time. This is especially interesting because for certain non-trivial equations the operators A_i need not necessarily commute.

As a bonus, we also find that we can replace the resolvents $(\lambda + A_i)^{-1}$ by arbitrary M-bounded resolvent families, that means by resolvent families $\{R_i(\lambda);\ \lambda > 0\}$ with $M := \sup_{\lambda>0} ||\lambda R_i(\lambda)|| < \infty$.

This paper is work based on the results of my diploma thesis [4] that focuses on proving more general operator equations with fractional powers of the operators A and $(\lambda + A)^{-1}$ rather than on the formulation of the functional calculus in terms of the Stieltjes transform.

During the time I worked on it, I was funded by the *Studienstiftung des Deutschen Volkes* and by *Bell Laboratories*, New Jersey.

I want to thank Prof. U. Westphal for her support during my undergraduate studies in Hannover that prepared me for this problem. I also want to thank Prof. Weis (University of Karlsruhe) for some advice concerning the functional calculus and Mark Hansen (Bell Laboratories Statistics Department) for his help with the translation of this publication.

Some Notation

For $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $\vec{k} \in \mathbb{N}_0^n$ we write $\vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i \cdot y_i$, $D^{\vec{k}} = \partial_1^{k_1} \cdots \partial_n^{k_n}$, $\vec{k}! = k_1! \cdots k_n!$, $|\vec{k}| = k_1 + \cdots + k_n$, $\ln \vec{x} = (\ln x_1, \ldots, \ln x_n)$ and for $\vec{k} \in \mathbb{Z}^n$ $\vec{x}^{\vec{k}} = x_1^{k_1} \cdots x_n^{k_n}$. We write $\vec{x} \leq \vec{y}$ iff $x_i \leq y_i$ for $\forall i = 1, \ldots, n$. Furthermore we write $\mathbf{1} = (1, \ldots, 1)$ and $\mathbf{0} = (0, \ldots, 0)$ where the dimension is clear for the context, $\mathbb{R}_+ = (0, \infty)$ and $\overline{\mathbb{R}}_+ = [0, \infty)$.

2 Introduction

The classical Stieltjes transform of a function $f : (0, \infty) \to \mathbb{C}$ is defined by (1) if the integrals exist absolutely for $\forall a > 0$. In this case f is twice Laplace-transformable with

$$\begin{aligned}\mathcal{L}(\mathcal{L}(f))(a) &= \int_0^\infty ds\, e^{-as} \int_0^\infty dt\, e^{-st} f(t) = \int_0^\infty dt\, f(t) \int_0^\infty ds\, e^{-s(t+a)} \\ &= \int_0^\infty f(t)\, \frac{1}{t+a}\, dt = \mathcal{S}(f)(a)\end{aligned}$$

for $\forall a > 0$. Since we will make use of the analogous relation for the distributional Stieltjes transformation, let us take a closer look at the distributional Laplace transform, where we refer the reader to the books by L. Schwartz [5] and A.H. Zemanian [6].

A distribution $T \in \mathcal{D}(\mathbb{R}^n)'$ with support in $\overline{\mathbb{R}}_+^n$ is Laplace-transformable and its transform $\mathcal{L}(T)$ is defined on $\mathbb{R}_+^n$ if T is a tempered distribution, $T \in \mathcal{S}(\mathbb{R}^n)'$, where $\mathcal{S}(\mathbb{R}^n)'$ is the topological dual of the locally convex vector space

$$\mathcal{S}(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n);\ ||\, \vec{x}^{\vec{k}} D^{\vec{j}} \varphi(\vec{x})||_\infty < \infty \quad \forall \vec{k}, \vec{j} \in \mathbb{N}_0^n\},$$

equipped with the topology induced by the set of the seminorms $|||\varphi|||_{\vec{k},\vec{j}} = ||\vec{x}^{\vec{k}} D^{\vec{j}} \varphi(\vec{x})||_\infty,\ \ \vec{k}, \vec{j} \in \mathbb{N}_0^n$.

The Laplace transform of such a distribution T is now defined by

$$\mathcal{L}(T)(\vec{s}) := T(\lambda \cdot e^{-\vec{s}\cdot})$$

for all $\vec{s} \in \mathbb{R}_+^n$, where $\lambda(\vec{x}) = \prod_{k=1}^n \psi(x_k)$ ($\vec{x} \in \mathbb{R}^n$) for an arbitrary function ψ in $C^\infty(\mathbb{R})$ with support bounded below that equals 1 on an interval $[-\varepsilon, \infty)$. Its uniqueness theorem states that if $\mathcal{L}(T)(\vec{s}) = 0$ for $\forall \vec{s} \in \mathbb{R}_+^n$ then T is the zero-distribution.

An interesting subclass of $\mathcal{S}(\mathbb{R}^n)'$ is the class $\mathcal{D}_{L^1}(\mathbb{R}^n)'$ of all integrable distributions. It is defined as the topological dual of the locally convex vector space

$\dot{\mathcal{B}}(\mathbb{R}^n)$, where

$$\mathcal{B}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n);\ ||D^{\vec{k}}\varphi||_\infty < \infty \ \ \forall \vec{k} \in \mathbb{N}_0^n\} \qquad \text{and}$$
$$\dot{\mathcal{B}}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n);\ \lim_{\vec{x}\to\infty} D^{\vec{k}}\varphi(\vec{x}) = 0 \ \ \forall \vec{k} \in \mathbb{N}_0^n\} \ \subsetneq \mathcal{B}(\mathbb{R}^n).$$

Here $\dot{\mathcal{B}}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^n)$ are equipped with the topology induced by the set of seminorms $||D^{\vec{k}}\varphi||_\infty$, $\vec{k} \in \mathbb{N}_0^n$. The set of all test functions $\mathcal{D}(\mathbb{R}^n)$ is dense in $\dot{\mathcal{B}}(\mathbb{R}^n)$ but not in $\mathcal{B}(\mathbb{R}^n)$.

One of the most important properties of this class of distributions is its characterization as the class of all distributions of the form

$$T = \sum_{|\vec{k}|\le K} D^{\vec{k}} f_{\vec{k}}\ , \tag{5}$$

where $f_{\vec{k}} \in L_1(\mathbb{R}^n)$ for $\forall|\vec{k}| \le K$.

Remark 1 *If* $n = 1$ *and* $\operatorname{supp} T \subseteq \overline{\mathbb{R}}_+$ *then we can also assume that the functions* f_k *in* (5) *also have support in* $\overline{\mathbb{R}}_+$ *for* $\forall k = 0, \dots, K$.

The class of integrable distributions with support in $\overline{\mathbb{R}}_+$ is denoted by $\mathcal{D}_{L^1}(\overline{\mathbb{R}}_+)'$. A proof of this remark can be found in [8, p. 298, Prop. 0.5]. Although one might intuitively expect this remark to hold also in higher-dimensional spaces, this is not entirely clear. For the proof in [8] the compactness of the boundary of $\mathbb{R}_+$ is essential, so a generalization of this remark would require a completely new approach.

This representation theorem can then be used to extend integrable distributions to $\mathcal{B}(\mathbb{R}^n)$ by approximating a function $\varphi \in \mathcal{B}(\mathbb{R}^n)$ *pointwise* by a sequence of functions $\varphi_m \in \dot{\mathcal{B}}(\mathbb{R}^n)$ and then defining $T(\varphi) := \lim_{m\to\infty} T(\varphi_m)$. For more details see [5] or [7].

A characterization similar to (5) for our class of strongly Stieltjes-transformable distributions and an analogous way to extend these distributions to a larger test function space will be the content of section 3. The restriction of Remark 1 to the case $n = 1$ is the only reason why we will define the space $\mathcal{D}_S(\overline{\mathbb{R}}_+)'$ in section 4 only in dimension one.

3 Strongly Stieltjes-transformable Distributions

As mentioned in the introduction, we start with the definition of the distribution class $\mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ as the topogical dual of some test function space. At this point we still treat the general n-dimensional case. Let $n \in \mathbb{N}$ be fixed.

Definition 1 *We define the spaces $\mathcal{H}(\mathbb{R}^n_+)$ and $\dot{\mathcal{H}}(\mathbb{R}^n_+)$ by*

$$\mathcal{H}(\mathbb{R}^n_+) := \{\varphi \in C^\infty(\mathbb{R}^n_+);\ ||\vec{x}^{\vec{k}+\mathbf{1}} D^{\vec{k}}\varphi(\vec{x})||_\infty < \infty\ \forall \vec{k} \in \mathbb{N}_0^n\},$$
$$\dot{\mathcal{H}}(\mathbb{R}^n_+) := \{\varphi \in C^\infty(\mathbb{R}^n_+);\ \lim_{m\to\infty} \sup_{\vec{x}\in I_m} |\vec{x}^{\vec{k}+\mathbf{1}} D^{\vec{k}}\varphi(\vec{x})| = 0\ \forall \vec{k} \in \mathbb{N}_0^n\} \subsetneq \mathcal{H}(\mathbb{R}^n_+),$$

where $I_m := \mathbb{R}^n_+ \setminus [\frac{1}{m}, m]^n$. We equip these spaces with the locally convex topology induced by the set of seminorms $\{||\cdot||_{\vec{k}};\ \vec{k} \in \mathbb{N}_0^n\}$, where $||\varphi||_{\vec{k}} := ||\vec{x}^{\vec{k}+\mathbf{1}} D^{\vec{k}}\varphi(\vec{x})||_\infty$.

We call $\mathcal{D}_{S^*}(\mathbb{R}^n_+)' := \dot{\mathcal{H}}(\mathbb{R}^n_+)'$ (i.e. the topological dual of $\dot{\mathcal{H}}(\mathbb{R}^n_+)$) the space of all strongly Stieltjes-transformable distributions.

The seminorms $||\cdot||_{\vec{k}}$ are actually norms on $\mathcal{H}(\mathbb{R}^n_+)$ and $\dot{\mathcal{H}}(\mathbb{R}^n_+)$ since the derivatives $D^{\vec{k}}\varphi$ are guaranteed to vanish at infinity so that we can conclude $||\varphi||_{\vec{k}} = 0 \Rightarrow D^{\vec{k}}\varphi = 0 \Rightarrow \varphi = 0$.

It is easy to see that all test functions $\varphi \in \mathcal{D}(\mathbb{R}^n_+) := C_c^\infty(\mathbb{R}^n_+)$ are in $\dot{\mathcal{H}}(\mathbb{R}^n_+)$. The next lemma shows that $\mathcal{D}(\mathbb{R}^n_+)$ is dense in $\dot{\mathcal{H}}(\mathbb{R}^n_+)$, and that $\dot{\mathcal{H}}(\mathbb{R}^n_+)$ is a closed subspace of $\mathcal{H}(\mathbb{R}^n_+)$. It is a real subspace because for every $\vec{a} \in \overline{\mathbb{R}}^n_+$ the function

$$\varphi_{\vec{a}}(\vec{x}) := (\vec{x} + \vec{a})^{-\mathbf{1}} = \prod_{i=1}^n \frac{1}{x_i + a_i}$$

belongs to $\mathcal{H}(\mathbb{R}^n_+)$, but not to $\dot{\mathcal{H}}(\mathbb{R}^n_+)$. To get used to the notation, the reader may verify that

$$D_{\vec{x}}^{\vec{k}}\,\varphi_{\vec{a}}(\vec{x}) = (-1)^{|\vec{k}|}\,\vec{k}!\,(\vec{x} + \vec{a})^{-\vec{k}-\mathbf{1}} \tag{6}$$

which will be important later on.

Lemma 1
(i) $\overline{\mathcal{D}(\mathbb{R}^n_+)}^{\mathcal{H}(\mathbb{R}^n_+)} = \dot{\mathcal{H}}(\mathbb{R}^n_+)$.
(ii) For every $\varphi \in \mathcal{H}(\mathbb{R}^n_+)$ there is a sequence $(\varphi_m)_{m\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n_+)$ with the properties

a) $\forall \vec{k} \in \mathbb{N}_0^n\, \forall \vec{x} \in \mathbb{R}^n_+ :\ \lim_{m\to\infty} D^{\vec{k}}\varphi_m(\vec{x}) = D^{\vec{k}}\varphi(\vec{x})$

b) $\forall \vec{k} \in \mathbb{N}_0^n :\ \sup_{m\in\mathbb{N}} ||\varphi_m||_{\vec{k}} < \infty$

Proof: (i) The inclusion $\subseteq$ is easy to see: Let $(\varphi_l)_l$ be a sequence in $\mathcal{D}(\mathbb{R}^n_+)$ that converges in $\mathcal{H}(\mathbb{R}^n_+)$ to a function $\varphi \in \mathcal{H}(\mathbb{R}^n_+)$. We need to show that $\varphi \in \dot{\mathcal{H}}(\mathbb{R}^n_+)$. Therefore let $\vec{k} \in \mathbb{N}_0^n$. Then for every $\varepsilon > 0$ there is an $l \in \mathbb{N}$ with $||\varphi - \varphi_l||_{\vec{k}} < \varepsilon$. Let $m_0 \in \mathbb{N}$ with $\varphi_l(\vec{x}) = 0$ for $\forall \vec{x} \in I_{m_0}$. Then for $\forall m \geq m_0$ we have

$$\sup_{\vec{x}\in I_m} |\vec{x}^{\vec{k}+\mathbf{1}} D^{\vec{k}}\varphi(\vec{x})| = \sup_{\vec{x}\in I_m} |\vec{x}^{\vec{k}+\mathbf{1}} D^{\vec{k}}(\varphi - \varphi_l)(\vec{x})| \leq ||\varphi - \varphi_l||_{\vec{k}} < \varepsilon$$

This shows that $\lim_{m\to\infty}\sup_{\vec{x}\in I_m}|\vec{x}^{\,\vec{k}+\mathbf{1}}D^{\vec{k}}\varphi(\vec{x})| = 0$. Since this is true for every $\vec{k}$, we proved that $\varphi \in \dot{\mathcal{H}}(\mathbb{R}^n_+)$.

For the inclusion $\supseteq$ let $\varphi \in \dot{\mathcal{H}}(\mathbb{R}^n_+)$, and we have to find a sequence $(\varphi_m)_m$ in $\mathcal{D}(\mathbb{R}^n_+)$ with $\lim_{m\to\infty}||\varphi_m - \varphi||_{\vec{k}} = 0$ for all $\vec{k} \in \mathbb{N}^n_0$. To define the sequence $(\varphi_m)_m$, let $\psi \in \mathcal{D}(\mathbb{R}_+)$ be an arbitrary test function with $0 \le \psi \le 1$ that has the value 1 in a neighborhood of $x = 1$ and whose support is contained in $(0, 2)$. Now, for all $m \in \mathbb{N}$ we define $\psi_m \in \mathcal{D}(\mathbb{R}_+)$ by

$$\psi_m(\lambda) := \begin{cases} \psi(m\lambda), & 0 < \lambda < \frac{1}{m}, \\ 1, & \frac{1}{m} \le \lambda \le m, \\ \psi(\frac{1}{m}\lambda), & m < \lambda, \\ \Big(0, & 2m < \lambda\Big) \end{cases} \qquad \psi_m^{(j)}(\lambda) = \begin{cases} m^j\psi^{(j)}(m\lambda), & 0 < \lambda < \frac{1}{m}, \\ 0, & \frac{1}{m} \le \lambda \le m, \\ m^{-j}\psi^{(j)}(\frac{1}{m}\lambda), & m < \lambda, \\ \Big(0, & 2m < \lambda\Big) \end{cases}$$

for $\forall j \in \mathbb{N}$. We find that for every $j \in \mathbb{N}_0$ $\lambda^j\psi_m^{(j)}(\lambda)$ is bounded uniformly with respect to λ and m:

$$\forall j \in \mathbb{N}_0\,\forall m \in \mathbb{N}: \sup_{\lambda > m^{-1}}|\lambda^j\psi_m^{(j)}(\lambda)| \le (2m)^j\tfrac{1}{m^j}||\psi^{(j)}||_\infty,$$

similarly for $0 < \lambda < \frac{1}{m}$. Therefore, if we define $\varphi_m(\vec{x}) := \varphi(\vec{x})\cdot\prod_{i=1}^n\psi_m(x_i) \in \mathcal{D}(\mathbb{R}^n_+)$ and apply Leibniz's rule, we have for $\forall\vec{x} \in \mathbb{R}^n_+$ $\forall\vec{k} \in \mathbb{N}^n_0$ and some constants $c_{\vec{j},\vec{k}}$

$$\begin{aligned} |\vec{x}^{\,\vec{k}+\mathbf{1}}D^{\vec{k}}(\varphi_m - \varphi)(\vec{x})| &= \Big|\vec{x}^{\vec{k}+\mathbf{1}}D^{\vec{k}}\Big(\varphi(\vec{x})\cdot\Big(\prod_{i=1}^n\psi_m(x_i) - 1\Big)\Big)\Big| \\ &= \Big|\sum_{\mathbf{0}\le\vec{j}\le\vec{k}} c_{\vec{j},\vec{k}}\,\vec{x}^{\,\vec{j}}\Big(D^{\vec{j}}(\prod_{i=1}^n\psi_m(x_i) - 1)\Big)\vec{x}^{\,(\vec{k}-\vec{j})+\mathbf{1}}D^{\vec{k}-\vec{j}}\varphi(\vec{x})\Big| \\ &= \Big|\Big(\prod_{i=1}^n\psi_m(x_i) - 1\Big)\vec{x}^{\,\vec{k}+\mathbf{1}}D^{\vec{k}}\varphi(\vec{x}) \\ &\quad + \sum_{\mathbf{0}\ne\vec{j}\le\vec{k}} c_{\vec{j},\vec{k}}\Big(\prod_{i=1}^n x_i^{j_i}\psi_m^{(j_i)}(x_i)\Big)\vec{x}^{\,(\vec{k}-\vec{j})+\mathbf{1}}D^{\vec{k}-\vec{j}}\varphi(\vec{x})\Big| \\ &\le const\cdot\sum_{\mathbf{0}\le\vec{j}\le\vec{k}}|\vec{x}^{\,(\vec{k}-\vec{j})+\mathbf{1}}D^{\vec{k}-\vec{j}}\varphi(\vec{x})|. \end{aligned} \tag{7}$$

Since $\varphi_m(\vec{x}) - \varphi(\vec{x}) = 0$ for $\forall\vec{x} \in [\frac{1}{m}, m]^n$, we can now conclude

$$||\varphi_m - \varphi||_{\vec{k}} \le const\cdot\sum_{\mathbf{0}\le\vec{j}\le\vec{k}}\sup_{\vec{x}\in I_m}|\vec{x}^{(\vec{k}-\vec{j})+\mathbf{1}}D^{\vec{k}-\vec{j}}\varphi(\vec{x})| \to 0$$

with $m \to \infty$.

(ii) Using the same definition for an approximation (φ_m) of an arbitrary $\varphi \in \mathcal{H}(\mathbb{R}^n_+)$, property a) is easily checked while b) follows from inequality (7): $||\varphi_m||_{\vec{k}} \leq ||\varphi||_{\vec{k}} + const \cdot \sum_{\mathbf{0}\leq\vec{j}\leq\vec{k}} ||\varphi||_{\vec{k}-\vec{j}}$. $\square$

The fact that $\mathcal{D}(\mathbb{R}^n_+)$ is dense in $\dot{\mathcal{H}}(\mathbb{R}^n_+)$ also justifies to call $\mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ a class of distributions: Since by Lemma 1 (i) a mapping $S \in \mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ is uniquely determined by its values in $\mathcal{D}(\mathbb{R}^n_+)$ and since for $\forall\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have $\varphi|_{\mathbb{R}^n_+} \in \dot{\mathcal{H}}(\mathbb{R}^n_+)$, we can identify every mapping $S \in \mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ with the distribution $\tilde{S} \in \mathcal{D}(\mathbb{R}^n)'$, $\tilde{S}(\varphi) := S(\varphi|_{\mathbb{R}^n_+})$ for $\forall\varphi \in \mathcal{D}(\mathbb{R}^n)$. A distribution $S \in \mathcal{D}(\mathbb{R}^n)'$ is then strongly Stieltjes-transformable iff $\operatorname{supp} S \subseteq \overline{\mathbb{R}}^n_+$ and S is bounded on $\mathcal{D}(\mathbb{R}^n)$ with respect to the topology on $\dot{\mathcal{H}}(\mathbb{R}^n_+)$.

Let us now show a representation theorem for $\mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ that is similar to the one for integrable distributions. Many of the following steps will be based on such representations.

Theorem 1 (Characterization of $\mathcal{D}_{S^*}(\mathbb{R}^n_+)'$)

(i) The spaces $\dot{\mathcal{B}}(\mathbb{R}^n)$ and $\dot{\mathcal{H}}(\mathbb{R}^n_+)$ as well as the spaces $\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{H}(\mathbb{R}^n_+)$ are topologically isomorphic.

(ii) A distribution $S \in \mathcal{D}(\mathbb{R}^n_+)'$ is strongly Stieltjes-transformable if and only if S has the form

$$S = \sum_{|\vec{k}|\leq K} D^{\vec{k}} \mu_{\vec{k}} \tag{8}$$

for some measures $\mu_{\vec{k}}$ on $\mathbb{R}^n_+$, $|\vec{k}| \leq K$, with

$$\int_{\mathbb{R}^n_+} \vec{x}^{\,-\vec{k}-\mathbf{1}}\, d|\mu_{\vec{k}}|(\vec{x}) < \infty. \tag{9}$$

In this case we can assume that the measures are given by measurable functions $f_{\vec{k}}$ with

$$\int_{\mathbb{R}^n_+} \vec{x}^{\,-\vec{k}-\mathbf{1}} |f_{\vec{k}}(\vec{x})| d\vec{x} < \infty. \tag{10}$$

(iii) If $n = 1$ and $\operatorname{supp} S \subseteq [1, \infty)$ then we can assume that also $\operatorname{supp} f_k \subseteq [1, \infty)$ for all $k = 0, \ldots, K$.

Proof: (i) Define the operator $\Pi : \dot{\mathcal{B}}(\mathbb{R}^n) \to \dot{\mathcal{H}}(\mathbb{R}^n_+)$ by $(\Pi\psi)(\vec{x}) := \vec{x}^{\,-\mathbf{1}}\psi(\ln \vec{x})$ $\forall\psi \in \dot{\mathcal{B}}(\mathbb{R}^n)$. Let us show that this operator really leads into $\dot{\mathcal{H}}(\mathbb{R}^n_+)$ and is bijective with

$$(\Pi^{-1}\varphi)(\vec{y}) = e^{\mathbf{1}\cdot\vec{y}}\varphi(e^{y_1}, \ldots, e^{y_n}) \qquad \forall\varphi \in \dot{\mathcal{H}}(\mathbb{R}^n_+). \tag{11}$$

First, for every $\psi \in \dot{\mathcal{B}}(\mathbb{R}^n)$ and every $\vec{k} \in \mathbb{N}^n_0$ we have

$$D^{\vec{k}}(\Pi\psi)(\vec{x}) \in \Big[\{\vec{x}^{\,-(\vec{k}+\mathbf{1})}(D^{\vec{j}}\psi)(\ln \vec{x});\ \mathbf{0} \leq \vec{j} \leq \vec{k}\}\Big],$$

where $[\dots]$ denotes the set of all finite linear combinations of the functions inside the brackets. Indeed, this can easily be shown by induction over $|\vec{k}|$ with the induction step following from

$$\partial_i \vec{x}^{\,-(\vec{k}+\mathbf{1})}(D^{\vec{j}}\psi)(\ln\vec{x}) = \Big(-(k_i+1)(D^{\vec{j}}\psi)(\ln\vec{x}) + (\partial_i D^{\vec{j}}\psi)(\ln\vec{x})\Big)\tfrac{1}{x_i}\vec{x}^{\,-(\vec{k}+\mathbf{1})}.$$

Therefore $\vec{x}^{\vec{k}+\mathbf{1}}D^{\vec{k}}(\Pi\psi)(\vec{x})$ is a linear combination of functions $(D^{\vec{j}}\psi)(\ln\vec{x})$, $\mathbf{0}\le\vec{j}\le\vec{k}$. This shows first that

$$||\Pi\psi||_{\vec{k},\mathcal{H}} \le C\sum_{\mathbf{0}\le\vec{j}\le\vec{k}}||D^{\vec{j}}\psi||_\infty = C\sum_{\mathbf{0}\le\vec{j}\le\vec{k}}||\psi||_{\vec{j},\mathcal{B}}$$

so that Π is bounded, and second that $\Pi\psi\in\dot{\mathcal{H}}(\mathbb{R}^n_+)$ if $\psi\in\dot{\mathcal{B}}(\mathbb{R}^n)$.

Now clearly Π is one-to-one with its inverse given by (11), and we can use this formula to prove in a similar way that for every $\varphi\in\dot{\mathcal{H}}(\mathbb{R}^n_+)$ and every $\vec{k}\in\mathbb{N}_0^n$ we have

$$D^{\vec{k}}(\Pi^{-1}\varphi)(\vec{y}) \in \Big[\big\{e^{(\vec{j}+\mathbf{1})\cdot\vec{y}}(D^{\vec{j}}\varphi)(e^{y_1},\dots,e^{y_n});\ \mathbf{0}\le\vec{j}\le\vec{k}\big\}\Big]. \tag{12}$$

Indeed, this follows easily by induction on $|\vec{k}|$ from

$$\begin{aligned}\partial_i e^{(\vec{j}+\mathbf{1})\cdot\vec{y}}(D^{\vec{j}}\varphi)(e^{y_1},\dots,e^{y_n}) &= (j_i+1)\cdot e^{(\vec{j}+\mathbf{1})\cdot\vec{y}}(D^{\vec{j}}\varphi)(e^{y_1},\dots,e^{y_n})\\ &\quad + e^{(\vec{j}+\mathbf{1})\cdot\vec{y}}\cdot e^{y_i}(\partial_i D^{\vec{j}}\varphi)(e^{y_1},\dots,e^{y_n}).\end{aligned}$$

Now (12) shows first that

$$||\Pi^{-1}\varphi||_{\vec{k},\mathcal{B}} = ||D^{\vec{k}}\Pi^{-1}\varphi||_\infty \le C\sum_{\mathbf{0}\le\vec{j}\le\vec{k}}\sup_{\vec{x}\in\mathbb{R}^n_+}|\vec{x}^{-(\vec{j}+\mathbf{1})}D^{\vec{j}}\varphi(\vec{x})| = C\sum_{\mathbf{0}\le\vec{j}\le\vec{k}}||\varphi||_{\vec{j},\mathcal{H}},$$

so that Π^{-1} is bounded, and second that $\Pi^{-1}\varphi\in\dot{\mathcal{B}}(\mathbb{R}^n)$ if $\varphi\in\dot{\mathcal{H}}(\mathbb{R}^n_+)$.

So Π is an isomorphism from $\dot{\mathcal{B}}(\mathbb{R}^n)$ to $\dot{\mathcal{H}}(\mathbb{R}^n_+)$. The same proof also shows that $\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{H}(\mathbb{R}^n_+)$ are isomorphic.

(ii) One direction is easy: If S has the form (8), then

$$|S(\varphi)| \le \sum_{|\vec{k}|\le K}\int_{\mathbb{R}^n_+}|D^{\vec{k}}\varphi(\vec{x})|\,d|\mu_{\vec{k}}|(\vec{x}) \le \sum_{|\vec{k}|\le K}\int_{\mathbb{R}^n_+}\vec{x}^{\,-(\vec{k}+\mathbf{1})}\,d|\mu_{\vec{k}}|(\vec{x})\cdot||\varphi||_{\vec{k}}$$

for every $\varphi\in\mathcal{D}(\mathbb{R}^n_+)$, so S is in $\mathcal{D}_{S^*}(\mathbb{R}^n_+)'$.

For the other direction, let $S\in\mathcal{D}_{S^*}(\mathbb{R}^n_+)'$. Define the linear functional $T:\dot{\mathcal{B}}(\mathbb{R}^n)\to\mathbb{C}$ by $T(\psi):=S(\Pi\psi)$ for all $\psi\in\dot{\mathcal{B}}(\mathbb{R}^n)$. Since $T=S\circ\Pi$ is the composition of two bounded operators, T is bounded and therefore an integrable distribution. Consequently, there exist functions $g_{\vec{j}} \in L_1(\mathbb{R}^n)$ so that $T = \sum_{|\vec{j}|\leq K} D^{\vec{j}} g_{\vec{j}}$. So for all $\varphi \in \dot{\mathcal{H}}(\mathbb{R}^n_+)$ we have

$$\begin{aligned} S(\varphi) &= T(\Pi^{-1}\varphi) = \sum_{|\vec{j}|\leq K} (-1)^{|\vec{j}|} \int_{\mathbb{R}^n} g_{\vec{j}}(\vec{y}) D^{\vec{j}}(\Pi^{-1}\varphi)(\vec{y})\, d\vec{y} \\ &\overset{(12)}{=} \sum_{|\vec{j}|\leq K} \sum_{|\vec{k}|\leq K} d_{\vec{j},\vec{k}} \int_{\mathbb{R}^n} g_{\vec{j}}(\vec{y}) e^{(\vec{k}+\mathbf{1})\cdot\vec{y}} (D^{\vec{k}}\varphi)(e^{y_1},\dots,e^{y_n})\, d\vec{y} \\ &= \sum_{|\vec{j}|\leq K} \sum_{|\vec{k}|\leq K} d_{\vec{j},\vec{k}} \int_{\mathbb{R}^n_+} g_{\vec{j}}(\ln\vec{x}) \vec{x}^{\vec{k}} (D^{\vec{k}}\varphi)(\vec{x})\, d\vec{x} \end{aligned}$$

for some constants $d_{\vec{j},\vec{k}}$. Here we made the substitutions $x_i = e^{y_i}$, $\frac{dx_i}{x_i} = dy_i$. Now we see that S has the form (8) with

$$f_{\vec{k}}(\vec{x}) := (-1)^{|\vec{k}|} \sum_{|\vec{j}|\leq K} d_{\vec{j},\vec{k}}\, g_{\vec{j}}(\ln\vec{x})\, \vec{x}^{\vec{k}}.$$

The functions $f_{\vec{k}}$ fulfill the estimate (10) since

$$\int_{\mathbb{R}^n_+} |g_{\vec{j}}(\ln\vec{x})\vec{x}^{\vec{k}}|\, \vec{x}^{-(\vec{k}+\mathbf{1})} d\vec{x} = \int_{\mathbb{R}^n_+} |g_{\vec{j}}(\ln\vec{x})|\, \vec{x}^{-\mathbf{1}} d\vec{x} = \int_{\mathbb{R}^n} |g_{\vec{j}}(\vec{y})| d\vec{y} < \infty.$$

(iii) If $n = 1$ and $\operatorname{supp} S \subseteq [1,\infty)$ then $\operatorname{supp} T \subseteq [0,\infty)$, and we can assume that the functions g_k also have support in $[0,\infty)$ by Remark 1, so that the functions f_k have support in $[1,\infty)$. □

The Hahn-Banach theorem tells us that every $S \in \mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ can be extended to a bounded functional on $\mathcal{H}(\mathbb{R}^n_+)$, but this extension is not unique since $\dot{\mathcal{H}}(\mathbb{R}^n_+)$ is not dense in $\mathcal{H}(\mathbb{R}^n_+)$ by Lemma 1 and the remarks preceding it. We use the approximation $(\varphi_m)_m$ from Lemma 1 (ii) to define a specific extension explicitly:

Definition 2 *For $S \in \mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ and $\varphi \in \mathcal{H}(\mathbb{R}^n_+)$ we define*

$$S(\varphi) := \lim_{m\to\infty} S(\varphi_m) \tag{13}$$

$$= \sum_{|\vec{k}|\leq K} \int_{\mathbb{R}^n_+} D^{\vec{k}}\varphi(\vec{x})\, d\mu_{\vec{k}}(\vec{x}), \tag{14}$$

where $(\varphi_m)_m$ is any approximation of φ as in Lemma 1 (ii) and the measures $\mu_{\vec{k}}$ are those of any representation (8) *of S.*

The fact that (13)=(14) is proven by interchanging limit and integral, justified by the estimates in Lemma 1 (ii) b). Representation (14) shows that the limit (13) exists, that it is independent of the particular choice of the sequence $(\varphi_m)_m$, that this definition extends S and that the extended functional is bounded on $\mathcal{H}(\mathbb{R}^n_+)$. Representation (13) shows that $S(\varphi)$ does not depend on the choice of the measures $\mu_{\vec{k}}$.

From now on, whenever we write $S(\varphi)$ for a distribution $S \in \mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ and a function $\varphi \in \mathcal{H}(\mathbb{R}^n_+)$, we identify S with its extension to $\mathcal{H}(\mathbb{R}^n_+)$ from Definition 2.

Alternatively, one could have used the isomorphism Π from Theorem 1 to define the extension via $S(\varphi) := (S \circ \Pi)(\Pi^{-1}\varphi)$ for $\forall \varphi \in \mathcal{H}(\mathbb{R}^n_+)$, where $S \circ \Pi \in \dot{\mathcal{B}}(\mathbb{R}^n)'$ is identified with its extension to $\mathcal{B}(\mathbb{R}^n)$. Π can also be used to prove Lemma 1 (ii). The approach we chose in this section shows nicely the similarities between the extension of integrable distributions to $\mathcal{B}(\mathbb{R}^n)$ and of strongly Stieltjes-transformable distributions to $\mathcal{H}(\mathbb{R}^n_+)$.

4 Stieltjes-transformable Distributions

In this section let $n = 1$. We want to enlarge the class $\mathcal{D}_{S^*}(\mathbb{R}_+)'$ in a way that also measures with $\int_0^\infty \frac{1}{\lambda+a}\, d|\mu|(\lambda) < \infty$ for $\forall a > 0$ but with $\int_0^\infty \lambda^{-1} d|\mu|(\lambda) = \infty$ are included.

Note that we have to restrict ourselves to the one-dimensional case since for $n > 1$ we do not have any information about the support of the functions $f_{\vec{k}}$. This also explains why we could not start with these weaker assumptions right away because we want to develop a multidimensional functional calculus.

Definition 3 *A distribution $S \in \mathcal{D}(\mathbb{R})'$ is called Stieltjes-transformable if*

(i) $\operatorname{supp} S \subseteq \overline{\mathbb{R}}_+$ *and*

(ii) $S \circ \tau \in \mathcal{D}_{S^*}(\mathbb{R}_+)'$,

where τ denotes the translation $(\tau\varphi)(x) = \varphi(x+1)$. We denote the class of all Stieltjes-transformable distributions by $\mathcal{D}_S(\overline{\mathbb{R}}_+)'$.

Theorem 2 (Characterization of $\mathcal{D}_S(\overline{\mathbb{R}}_+)'$) *A distribution $S \in \mathcal{D}(\mathbb{R})'$ is Stieltjes-transformable if and only if S has the form*

$$S = \sum_{k=0}^{K} D^k \mu_k \tag{15}$$

for some measures μ_k on $\overline{\mathbb{R}}_+$, $0 \le k \le K$, with

$$\int_{\overline{\mathbb{R}}_+} (x+1)^{-k-1}\, d|\mu_k|(x) < \infty. \tag{16}$$

In this case we can assume that the measures μ_k are given by measurable *functions f_k with*

$$\int_{\mathbb{R}_+} (x+1)^{-k-1} |f_k(x)| dx < \infty. \tag{17}$$

PROOF: If S has the form (15) then it has support in $\overline{\mathbb{R}}_+$, and the translated distribution $S \circ \tau$ is represented by $\sum_{k=0}^{K} D^k (\tau^{-1}\mu_k)$ where $\tau^{-1}\mu_k$ denotes the shifted measure $(\tau^{-1}\mu_k)(A) = \mu_k(A-1)$. Since because of (16) the measures $\tau^{-1}\mu_k$ fulfill (9), we have $S \circ \tau \in \mathcal{D}_{S^*}(\mathbb{R}_+)'$ by Theorem 1.

If S is Stieltjes-transformable then $S \circ \tau$ is in $\mathcal{D}_{S^*}(\mathbb{R}_+)'$ and therefore has the form $\sum_{k=0}^{K} D^k g_k$ for some functions that fulfill (10). Defining $f_k := \tau g_k$, (15) holds for $d\mu_k(x) = f_k(x)\,dx$. Since the support of $S \circ \tau$ is contained in $[0, \infty)$, we can assume that the functions g_k also have support in $[0, \infty)$ by Theorem 1 (iii), so that property (10) for g_k translates into property (17) for f_k. $\square$

Corollary 1

(i) $\mathcal{D}_{L^1}(\overline{\mathbb{R}}_+)' \subset \mathcal{D}_S(\overline{\mathbb{R}}_+)'$
(ii) $\mathcal{D}_{S^*}(\mathbb{R}_+)' \subset \mathcal{D}_S(\overline{\mathbb{R}}_+)'$

PROOF: For (i), Remark 1 and Theorem 2; for (ii), Theorems 1 and 2. $\square$

Using the definition of $\mathcal{D}_S(\overline{\mathbb{R}}_+)'$, we can now extend every distribution $S \in \mathcal{D}_S(\overline{\mathbb{R}}_+)'$ from $\mathcal{D}(\mathbb{R})$ to a larger class of functions that includes the core of the Stieltjes transformation, φ_a ($a \in \mathbb{R}_+$).

Definition 4 *Let*

$$\mathcal{G}(\overline{\mathbb{R}}_+) := \{\varphi \in C^\infty(\overline{\mathbb{R}}_+);\ \exists \psi_\varphi \in \mathcal{H}(\mathbb{R}_+) : \varphi = (\tau\psi_\varphi)|_{\overline{\mathbb{R}}_+}\}.$$

We define for every $\varphi \in \mathcal{G}(\overline{\mathbb{R}}_+)$

$$S(\varphi) := (S \circ \tau)(\psi_\varphi) \tag{18}$$

$$= \sum_{k=0}^{K} (-1)^k \int_{\overline{\mathbb{R}}_+} D^k\varphi(x)\, d\mu_k(x), \tag{19}$$

where the measures μ_k are those of any representation of S as in Theorem 2.

The expressions (18) and (2) are equal because $(S\circ\tau)(\psi_\varphi) = S(\tau(\psi_\varphi)) = S(\varphi)$. The first expression shows that $S(\varphi)$ does not depend on the representation of S whereas the second expression shows that it does not depend on the choice of ψ_φ.

Note that with a technique similar to the one in the proof of Lemma 1 (ii) one can show that for every function $\varphi \in C^\infty((-\varepsilon, \infty))$ with $\forall k \in \mathbb{N}_0 : \sup_{x>0}|(x+1)^{k+1}\partial_k\varphi(x)| < \infty$ we have $\varphi|_{\overline{\mathbb{R}}_+} \in \mathcal{G}(\overline{\mathbb{R}}_+)$. Therefore the functions $\varphi_a(x) = (x+a)^{-1}$ ($x \in \overline{\mathbb{R}}_+$, $a \in \mathbb{R}_+$) are in $\mathcal{G}(\overline{\mathbb{R}}_+)$.

5 Stieltjes Transformation

Definition 5 *The Stieltjes transform of a distribution* $S \in \mathcal{D}_S(\overline{\mathbb{R}}_+)'$ *(for* $n = 1$*) or* $S \in \mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ *(for* $\forall n \in \mathbb{N}$*) is defined by*

$$\begin{aligned}\mathcal{S}(S)(\vec{a}) &:= S(\varphi_{\vec{a}})\\ &= \sum_{|\vec{k}|\leq K} \vec{k}! \int_{\overline{\mathbb{R}}^n_+} (\vec{x}+\vec{a})^{-\vec{k}-\mathbf{1}} d\mu_{\vec{k}}(\vec{x})\end{aligned}$$

for all $(\vec{a} \in \mathbb{R}^n_+)$*. The second expression holds for any representation* (8) *or* (15) *of S.*

Theorem 3 (Uniqueness theorem) *If* $S \in \mathcal{D}_S(\overline{\mathbb{R}}_+)'$ *or* $S \in \mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ *and* $\mathcal{S}(S)(\vec{a}) = 0$ *for* $\forall \vec{a} \in \mathbb{R}^n_+$*, then S is the zero-distribution.*

Proof: We show the case $S \in \mathcal{D}_{S^*}(\mathbb{R}^n_+)'$. Theorem 1 shows that S is a tempered distribution, and its Laplace-transform is

$$\mathcal{L}(S)(\vec{s}) = S(e^{-\vec{s}\cdot}) = \sum_{|\vec{k}|\leq K} \vec{s}^{\vec{k}} \int_{\overline{\mathbb{R}}^n_+} e^{-\vec{s}\vec{x}} d\mu_{\vec{k}}(\vec{x}) \qquad (\vec{s} \in \mathbb{R}^n_+).$$

This function is again Laplace-transformable in the classical sense with

$$\begin{aligned}\mathcal{L}(\mathcal{L}(S))(\vec{a}) &= \sum_{|\vec{k}|\leq K} \int_{\overline{\mathbb{R}}^n_+} d\vec{s}\, e^{-\vec{a}\vec{s}}\, \vec{s}^{\vec{k}} \int_{\overline{\mathbb{R}}^n_+} d\mu_{\vec{k}}(\vec{x}) e^{-\vec{s}\vec{x}}\\ &= \sum_{|\vec{k}|\leq K} \int_{\overline{\mathbb{R}}^n_+} d\mu_{\vec{k}}(\vec{x}) \int_{\overline{\mathbb{R}}^n_+} d\vec{s}\, \vec{s}^{\vec{k}}\, e^{-(\vec{x}+\vec{a})\vec{s}}\\ &= \sum_{|\vec{k}|\leq K} \vec{k}! \int_{\overline{\mathbb{R}}^n_+} d\mu_{\vec{k}}(\vec{x}) (\vec{x}+\vec{a})^{-\vec{k}-\mathbf{1}}\\ &= \mathcal{S}(S)(\vec{a}) = 0\end{aligned}$$

for $\forall \vec{a} \in \mathbb{R}^n_+$. The uniqueness theorem for the Laplace-transformation applied twice now first says that $\mathcal{L}(S) \equiv 0$ and then that $S = 0$. The case $S \in \mathcal{D}_S(\overline{\mathbb{R}}_+)'$ is proven analogously. □

Corollary 2 *Every* $S \in \mathcal{D}_S(\overline{\mathbb{R}}_+)'$ *and every* $S \in \mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ *is a tempered distribution that is twice Laplace-transformable with* $\mathcal{S}(S) = \mathcal{L}(\mathcal{L}(S))$*.*

6 Stieltjes Convolution

For the definition of the convolution we need the operators Δ_n that will be defined now.

Definition 6 *(i) Define the linear operator* $\Delta : \mathcal{H}(\mathbb{R}_+) \to \mathcal{H}(\mathbb{R}_+^2)$ *by*

$$\begin{aligned}(\Delta\varphi)(x,y) &:= \begin{cases} -\frac{\varphi(x)-\varphi(y)}{x-y} & \text{for } x \neq y, \\ -\varphi'(x) & \text{for } x = y. \end{cases} \\ &= -\int_0^1 \varphi'(tx+(1-t)y)\, dt.\end{aligned}$$

(ii) Define the linear operators $\Delta_n : \mathcal{H}(\mathbb{R}_+^n) \to \mathcal{H}((\mathbb{R}_+^n)^2)$ *for* $\forall n \in \mathbb{N}$ *by*

$$(\Delta_n\varphi)(\vec{\lambda}_1, \vec{\lambda}_2) = (\Delta_{x_1\to(\lambda_{1,1},\lambda_{2,1})} \circ \cdots \circ \Delta_{x_n\to(\lambda_{1,n},\lambda_{2,n})})(\varphi(x_1,\ldots,x_n)).$$

Here the notation $\Delta_{x\to(y,z)}\varphi(x,\ldots)$ means that Δ treats φ as a function of x and "creates" the variables y and z. This means that Δ_n is the composition of n operators Δ, each acting on a different variable x_i, $i = 1,\ldots,n$, and increasing the number of variables by one.

Lemma 2 *The operators in Definition 6 are well-defined and bounded.*

Proof: We will show that for $\forall p \in \mathbb{N}$, $\forall i \in \{1,\ldots,p\}$ and $\forall\varphi \in \mathcal{H}(\mathbb{R}_+^p)$, $\Delta_{x_i\to(\lambda_{1,i},\lambda_{2,i})}\varphi$ is well-defined and in $\mathcal{H}(\mathbb{R}_+^{p+1})$. We can assume that $i = p$. It can easily be checked that for fixed $x_1,\ldots,x_{p-1}$ $\varphi(x_1,\ldots,x_p)$ is in $\mathcal{H}(\mathbb{R}_+)$, so the expression is well-defined. Now let $\vec{k} \in \mathbb{N}_0^{p+1}$ and $\vec{x} \in \mathbb{R}_+^{p+1}$. Let $\vec{k}' := (k_1,\ldots,k_{p-1},k_p+k_{p+1}+1)$. Then we have

$$\begin{aligned}
&|\vec{x}^{\vec{k}+\mathbf{1}} D_{\vec{x}}^{\vec{k}}(\Delta_{x_p\to(x_p,x_{p+1})}\varphi)(\vec{x})| \\
&= \left|\vec{x}^{\vec{k}+\mathbf{1}} D_{\vec{x}}^{\vec{k}} \int_0^1 (\partial_p\varphi)(x_1,\ldots,x_{p-1},tx_p+(1-t)x_{p+1})\, dt\right| \\
&= \left|\vec{x}^{\vec{k}+\mathbf{1}} \int_0^1 t^{k_p}(1-t)^{k_{p+1}}(D^{\vec{k}'}\varphi)(x_1,\ldots,x_{p-1},tx_p+(1-t)x_{p+1})\, dt\right| \\
&\leq ||\vec{x}^{\vec{k}'+\mathbf{1}} D^{\vec{k}'}\varphi(\vec{x})||_\infty \cdot \\
&\qquad \cdot \left|\vec{x}^{\vec{k}+\mathbf{1}} \int_0^1 t^{k_p}(1-t)^{k_{p+1}}(x_1,\ldots,x_{p-1},tx_p+(1-t)x_{p+1})^{-\vec{k}'-\mathbf{1}}\, dt\right| \\
&= ||\varphi||_{\vec{k}'} \cdot \left|x_p^{k_p+1}x_{p+1}^{k_{p+1}+1} \int_0^1 t^{k_p}(1-t)^{k_{p+1}}[tx_p+(1-t)x_{p+1}]^{-(k_p+k_{p+1}+1)-1}dt\right| \\
&= ||\varphi||_{\vec{k}'} \cdot x_p^{k_p+1}x_{p+1}^{k_{p+1}+1}\left|\partial_p^{k_p}\partial_{p+1}^{k_{p+1}}(k_p+k_{p+1}+1)!^{-1}\int_0^1 [tx_p+(1-t)x_{p+1}]^{-2}dt\right| \\
&= ||\varphi||_{\vec{k}'} \cdot x_p^{k_p+1}x_{p+1}^{k_{p+1}+1} \cdot \\
&\qquad \cdot \left|\partial_p^{k_p}\partial_{p+1}^{k_{p+1}}(k_p+k_{p+1}+1)!^{-1}\tfrac{1}{x_p-x_{p+1}}\left[(tx_p+(1-t)x_{p+1})^{-1}\right]_{t=0}^{t=1}\right| \\
&= ||\varphi||_{\vec{k}'} \cdot x_p^{k_p+1}x_{p+1}^{k_{p+1}+1}\left|\partial_p^{k_p}\partial_{p+1}^{k_{p+1}}(k_p+k_{p+1}+1)!^{-1}\tfrac{1}{x_px_{p+1}}\right| \\
&= ||\varphi||_{\vec{k}'} \cdot \tfrac{k_p!\,k_{p+1}!}{(k_p+k_{p+1}+1)!}.
\end{aligned}$$

Consequently $\Delta_{x_p\to(x_p,x_{p+1})}\varphi \in \mathcal{H}(\mathbb{R}_+^{p+1})$, and we see that $\Delta_{x_p\to(x_p,x_{p+1})}$ is bounded. $\square$

The following Lemma is the main property of these operators in the context of the Stieltjes transformation, which is the key to the definition of the Stieltjes convolution: It shows that the product of two cores of the Stieltjes transformation can be written as a bounded linear operation acting on only one core:

Lemma 3 *For $\forall n \in \mathbb{N}$ and $\forall \vec{a} \in \overline{\mathbb{R}}_+^n$ we have*

$$\Big(\Delta_n \varphi_{\vec{a}}\Big)(\vec{\lambda}_1, \vec{\lambda}_2) = \varphi_{\vec{a}}(\vec{\lambda}_1)\cdot\varphi_{\vec{a}}(\vec{\lambda}_2) \qquad (\vec{\lambda}_1, \vec{\lambda}_2 \in \mathbb{R}_+^n).$$

PROOF: The resolvent equation $-\frac{\frac{1}{x+a}-\frac{1}{y+a}}{x-y} = \frac{1}{x+a}\frac{1}{y+a}$ proves the case $n=1$, the general case follows easily by induction. $\square$

We now define the Stieltjes convolution, first only on $\mathcal{D}_{S^*}(\mathbb{R}_+^n)'$.

Definition 7 *For two distributions $S, T \in \mathcal{D}_{S^*}(\mathbb{R}_+^n)'$ the convolution $S * T \in \mathcal{D}_{S^*}(\mathbb{R}_+^n)'$ is defined by*

$$S * T := (S \otimes T)\circ \Delta_n,$$

that means

$$(S*T)(\varphi) = (S_{\vec{\lambda}_1}\circ T_{\vec{\lambda}_2})\Big((\Delta_n\varphi)(\vec{\lambda}_1, \vec{\lambda}_2)\Big) \qquad (\varphi \in \dot{\mathcal{H}}(\mathbb{R}_+^n)).$$

Lemma 4 *(i) The Stieltjes convolution is well-defined, commutative and associative.*

(ii) $\forall S, T \in \mathcal{D}_{S^*}(\mathbb{R}_+^n)'$: $\mathcal{S}(S*T) = \mathcal{S}(S)\cdot\mathcal{S}(T)$

(iii) We have the formula

$$(S*T)(\varphi) = \sum_{|\vec{k}_1|\le K_1}\sum_{|\vec{k}_2|\le K_2}(-1)^{|\vec{k}_1|+|\vec{k}_2|}\int_{\mathbb{R}_+^n} d\mu^1_{\vec{k}_1}(\vec{\lambda}_1)\int_{\mathbb{R}_+^n} d\mu^2_{\vec{k}_2}(\vec{\lambda}_2) \\ D^{\vec{k}_1}_{\vec{\lambda}_1} D^{\vec{k}_2}_{\vec{\lambda}_2}(\Delta_n\varphi)(\vec{\lambda}_1,\vec{\lambda}_2), \quad (20)$$

where the measures $\mu^i_{\vec{k}_i}$ are those from any representation for S and T from Theorem 1.

PROOF: Formula (20) is a direct consequence of the definition. The properties of the measures $\mu^i_{\vec{k}_i}$ and of $\Delta_n\varphi \in \mathcal{H}((\mathbb{R}_+^n)^2)$ ensure that the integrals exist absolutely, so the expression is well-defined. In the same way (replace $\Delta_n\varphi$ in (20) by φ to obtain an expression for $(S\otimes T)(\varphi)$) one can see that $S \otimes T$:

$\mathcal{H}(\mathbb{R}_+^{2n}) \to \mathbb{C}$ is bounded, so that $S * T$ is bounded as the composition of the two bounded operators $S \otimes T$ and Δ_n.

Commutativity can be shown by interchanging the order of summation, integration and differentiation in (20), respectively, and by using the symmetry of $\Delta_n \varphi$ in the variables $\vec{\lambda}_1$ and $\vec{\lambda}_2$.

From Definition 5, Equation (20), Lemma 3 and Equation (6) we conclude

$$\begin{aligned}
\mathcal{S}(S * T)(\vec{a}) &= (S * T)(\varphi_{\vec{a}}) \\
&= \sum_{|\vec{k}_1| \le K_1} \sum_{|\vec{k}_2| \le K_2} (-1)^{|\vec{k}_1| + |\vec{k}_2|} \int_{\mathbb{R}_+^n} d\mu^1_{\vec{k}_1}(\vec{\lambda}_1) \int_{\mathbb{R}_+^n} d\mu^2_{\vec{k}_2}(\vec{\lambda}_2) \\
&\qquad D^{\vec{k}_1}_{\vec{\lambda}_1} D^{\vec{k}_2}_{\vec{\lambda}_2} (\varphi_{\vec{a}}(\vec{\lambda}_1) \cdot \varphi_{\vec{a}}(\vec{\lambda}_2)) \\
&= \sum_{|\vec{k}_1| \le K_1} \sum_{|\vec{k}_2| \le K_2} \vec{k}_1! \, \vec{k}_2! \int_{\mathbb{R}_+^n} d\mu^1_{\vec{k}_1}(\vec{\lambda}_1) \int_{\mathbb{R}_+^n} d\mu^2_{\vec{k}_2}(\vec{\lambda}_2) \cdot \\
&\qquad \cdot (\vec{\lambda}_1 + \vec{a})^{-\vec{k}_1 - \mathbf{1}} (\vec{\lambda}_2 + \vec{a})^{-\vec{k}_2 - \mathbf{1}} \\
&= \prod_{i=1}^{2} \sum_{|\vec{k}_i| \le K_i} \vec{k}_i! \int_{\mathbb{R}_+^n} d\mu^i_{\vec{k}_i}(\vec{\lambda}_i) (\vec{\lambda}_i + \vec{a})^{-\vec{k}_i - \mathbf{1}} \\
&= \prod_{i=1}^{2} \sum_{|\vec{k}_i| \le K_i} \vec{k}_i! \int_{\mathbb{R}_+^n} d\mu^i_{\vec{k}_i}(\vec{\lambda}_i) \, D^{\vec{k}_i}_{\vec{\lambda}_i} \varphi_{\vec{a}}(\vec{\lambda}_i) \\
&= S(\varphi_{\vec{a}}) \cdot T(\varphi_{\vec{a}}) = \mathcal{S}(S)(\vec{a}) \cdot \mathcal{S}(T)(\vec{a}).
\end{aligned}$$

for every $\vec{a} \in \mathbb{R}_+^n$. Finally, associativity can now be shown by computing

$$\begin{aligned}
\mathcal{S}(S_1 * (S_2 * S_3)) &= \mathcal{S}(S_1) \cdot \mathcal{S}(S_2 * S_3) = \mathcal{S}(S_1) \cdot \mathcal{S}(S_2) \cdot \mathcal{S}(S_3) \\
&= \mathcal{S}(S_1 * S_2) \cdot \mathcal{S}(S_3) = \mathcal{S}((S_1 * S_2) * S_3)
\end{aligned}$$

and by applying the uniqueness theorem to conclude that

$$S_1 * (S_2 * S_3) = (S_1 * S_2) * S_3.$$

□

To extend the convolution to the space $\mathcal{D}_S(\overline{\mathbb{R}}_+)'$ (if $n = 1$), note that

(i) If $\varphi(x) = 0$ for all $x \in [1, \infty)$ then $\Delta\varphi(\vec{x}) = 0$ for all $\vec{x} \in [1, \infty)^2$.

(ii) Therefore, if $S, T \in \mathcal{D}_{S^*}(\mathbb{R}_+)'$ have support in $[1, \infty)$ then the same is true for $S * T$ by representation (20), since according to Theorem 1*(iii)* we may assume that the measures $\mu^i_{\vec{k}_i}$ in (20) have support in $[1, \infty)$ as well.

(iii) If $S, T \in \mathcal{D}_S(\overline{\mathbb{R}}_+)'$, then (ii) tells us that $(S \circ \tau) * (T \circ \tau)$ has support in $[1, \infty)$.

This shows that the distribution in the following Definition 8 is well-defined and in $\mathcal{D}_S(\overline{\mathbb{R}}_+)'$.

Definition 8 *For two distributions $S, T \in \mathcal{D}_S(\overline{\mathbb{R}}_+)'$ we define the convolution $S * T \in \mathcal{D}_S(\overline{\mathbb{R}}_+)'$ by*

$$(S * T)(\varphi) := \Big((S \circ \tau) * (T \circ \tau)\Big)(\tau^{-1}\varphi) \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

Lemma 5 *(i) The extended Stieltjes convolution in Definition 8 is commutative, associative.*

(ii) $\forall S, T \in \mathcal{D}_S(\overline{\mathbb{R}}_+)' : \ \mathcal{S}(S * T) = \mathcal{S}(S) \cdot \mathcal{S}(T)$

Proof: Commutativity follows directly from the definition and the commutativity of the convolution on $\mathcal{D}_{S^*}(\mathbb{R}_+)'$. For the last property we use representation (20) for the convolution on $\mathcal{D}_{S^*}(\mathbb{R}_+)'$ and derive easily the analogous relation for all functions $\varphi \in \mathcal{G}(\overline{\mathbb{R}}_+)$ and distributions $S, T \in \mathcal{D}_S(\overline{\mathbb{R}}_+)'$ with measures $\mu^i_{k_i}$ from Theorem 2. Following the lines of the proof in Lemma 4 we see that also in this general case we have $\mathcal{S}(S*T) = \mathcal{S}(S) \cdot \mathcal{S}(T)$ and therefore associativity. $\square$

The representation formula (20) for the convolution on $\mathcal{D}_S(\overline{\mathbb{R}}_+)'$ also shows that the convolution defined in Definition 8 coincides with the one from Definition 7 in the case that both distributions S and T are in $\mathcal{D}_{S^*}(\mathbb{R}_+)'$.

We now give a brief counterexample similar to the one mentioned in the introduction that shows that there are measures so that the product of their Stieltjes transforms is not the Stieltjes transform of any other measure.

Example 1 $\delta_0 * \delta_0 = D\delta_0$.

Proof: One can see that δ_0 is in $\mathcal{D}_S(\overline{\mathbb{R}}_+)'$ but not in $\mathcal{D}_{S^*}(\mathbb{R}_+)'$. Its Stieltjes transform is

$$\mathcal{S}(\delta_0)(a) = \int_{\overline{\mathbb{R}}_+} d\delta_0(\lambda) \frac{1}{\lambda + a} = \frac{1}{a}$$

for $\forall a > 0$. Therefore we find that

$$\mathcal{S}(\delta_0 * \delta_0)(a) = \mathcal{S}(\delta_0)(a) \cdot \mathcal{S}(\delta_0)(a) = \tfrac{1}{a^2} = 1! \int_{\overline{\mathbb{R}}_+} d\delta_0(\lambda) \Big(\tfrac{1}{\lambda + a}\Big)^{1+1} = \mathcal{S}(D\delta_0)(a)$$

for $\forall a > 0$, which shows that $\delta_0 * \delta_0 = D\delta_0$. $\square$

Finally, we want to state that the convolution algebra does not have a neutral element I since this would need to fulfill $S * I = S$ for $\forall S \in \mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ and therefore $\mathcal{S}(S) = \mathcal{S}(S) \cdot \mathcal{S}(I) \ \Rightarrow \ \mathcal{S}(I) \equiv 1$. This is not possible because every Stieltjes transform tends to 0 with $\vec{a} \to \infty$.

7 A Functional Calculus for Non-negative Operators

In the context of fractional powers of operators on a Banach space X, mainly two classes of operators are treated: The class of the negatives of infinitesimal generators of uniformly bounded strongly continuous semigroups of operators, and non-negative operators.

For the first one, a subclass of the second one, Laurent Schwartz created in [10] a very powerful functional calculus that assigns—for a fixed semigroup generator of that class—to every Laplace transform of an integrable distribution a bounded operator on X. One can loosely think of this functional calculus as the mapping

$$\sum_{k=1}^{N}\int_0^\infty e^{-at}\partial_t^k\,\mu_k(dt) \quad\mapsto\quad \sum_{k=1}^{N}\int_0^\infty e^{-At}A^k\,\mu_k(dt).$$

(Think of the derivative as acting on the exponential via partial integration, i.e. in the distributional sense.) This functional calculus was for example used by U. Westphal in [11] in the context of fractional powers of operators.

Inspired by this nice duality, we want to define a functional calculus that assigns—for a fixed non-negative operator—to every Stieltjes transform of a strongly Stieltjes-transformable distribution a bounded operator on X by

$$\sum_{k=1}^{N}\int_0^\infty \frac{1}{t+a}\partial_t^k\,\mu_k(dt) \quad\mapsto\quad \sum_{k=1}^{N}k!\int_0^\infty \left(\frac{1}{t+A}\right)^{(k+1)}\mu_k(dt),$$

as mentioned in SecTion 1. For non-negative operators, several other tools to prove operator equations exist: In 1960 Balakrishnan used in [9] a method to prove only the operator equations he needed, and in 1972 H.W. Hövel and U. Westphal ([1]) put this idea in the context of their classical Stieltjes convolution mentioned in Section 1. Both papers only dealt with the one-dimensional case. The methods in these two papers cover the results of Lemma 7 in the case that the distribution is a measure μ with $\int_{\mathbb{R}_+}\lambda^{-1}d|\mu|(\lambda) < \infty$, and of Theorem 4 when S_1 and S_2 are functions that fulfill certain estimates that guarantee that their convolution is again a function. In any way, with their method it is not possible to treat operator equations with operators $(\lambda + A)^{-n}$, $n \geq 2$.

Also in 1972 Hirsch used a technique in [12] that does not require the estimates for the functions as in [1]. Instead of the classical Stieltjes transformation his proof is based on the iterated application of the distributional Laplace transformation, and he implicitly used the Stieltjes convolution of two measures. His work was one of the main inspirations for this paper.

Another very powerful tool based on path integrals in the complex plane and the Cauchy integral theorem rather than transform methods is the more recently developed H^∞-calculus (see for example the papers by McIntosh [13], Cowling et al. [14] or the more recent one by Kalton and Weis [15], just to name a few). One of the main advantages of this approach is that it naturally leads to a spectral theorem, but it does not allow for non-commuting operators and purely real Banach spaces.

We will now use our approach to the distributional Stieltjes transform to define a functional calculus. Since we use the properties of the resolvents of A only, the calculus can be applied to all M-bounded resolvent families. A family $\{R(\lambda) : \ \lambda > 0\}$ of bounded operators on a Banach space X is called an M-bounded resolvent family if it fulfills the resolvent equation

$$R(\lambda_1) - R(\lambda_2) = (\lambda_2 - \lambda_1)R(\lambda_1)R(\lambda_2) \qquad \forall \lambda_1, \lambda_2 > 0$$

and has the property $\sup_{\lambda>0} ||\lambda R(\lambda)|| \leq M$. It is easy to check that $R(\cdot)$ is infinitely often differentiable with $\partial_k R(\cdot) = (-1)^k k! R(\cdot)^{k+1}$. Note that the resolvent families $\{(\lambda + A)^{-1};\ \lambda > 0\}$ of every non-negative operator is M-bounded.

We will restrict our functional calculus to those distributions in $\mathcal{D}_{S^*}(\mathbb{R}^n_+)'$ and expect it to return only bounded operators. An extension to the whole class $\mathcal{D}_S(\overline{\mathbb{R}}_+)'$ for $n = 1$ would probably also return unbounded operators and could provide an approach to fractional powers of operators with negative exponents. This conjecture is based on the observation that for every $\alpha > 0$ and an arbitrary $m \in \mathbb{N}$ with $m > \alpha$ the distribution $S_\alpha := C_{\alpha,m}^{-1} D^{m-1} \lambda^{m-\alpha-1}$, where $C_{\alpha,m} := (m-1)!^{-1} \int_0^\infty \lambda^{m-\alpha-1} (\frac{1}{\lambda+1})^m d\lambda$, is in $\mathcal{D}_S(\overline{\mathbb{R}}_+)' \setminus \mathcal{D}_{S^*}(\mathbb{R}_+)'$ with

$$\mathcal{S}(S_\alpha)(a) = C_{\alpha,m}^{-1} (m-1)! \int_0^\infty \lambda^{m-\alpha-1} \Big(\frac{1}{\lambda + a}\Big)^m d\lambda = a^{-\alpha}.$$

Definition 9 *Let $\{R_i(\lambda);\ \lambda > 0\}$ for every $i = 1, \dots, n$ be an M-bounded resolvent family on the Banach space X. We define the linear mapping $\mathcal{R}_R : \mathcal{D}_{S^*}(\mathbb{R}^n_+)' \to L(X)$ by*

$$\mathcal{R}_R(S) := \sum_{|\vec{k}| \leq K} \vec{k}! \int_{\mathbb{R}^n_+} \prod_{i=1}^n R_i(\lambda_i)^{k_i+1} d\mu_{\vec{k}}(\vec{\lambda}),$$

if S has the representation (8).

It is not clear yet that our definition is independent of the particular representation (8) of S. To prove this, we will first show Lemma 6 that tells us that the values $\langle \mathcal{R}_R(S)x, x^* \rangle$ do not depend on the representation from which the operator $\mathcal{R}_R(S)$ was derived.

Lemma 6 *If we define $\varphi_{x,x^*} \in \mathcal{H}(\mathbb{R}^n_+)$ by $\varphi_{x,x^*}(\vec{\lambda}) := \Big\langle (\prod_{i=1}^n R_i(\lambda_i))x, x^* \Big\rangle$*

$(\vec{\lambda} \in \mathbb{R}_+^n)$, *then we have for* $\forall x \in X$ *and* $\forall x^* \in X^*$

$$S(\varphi_{x,x^*}) = \langle \mathcal{R}_R(S)x, x^* \rangle.$$

PROOF: The function φ_{x,x^*} is in $\mathcal{H}(\mathbb{R}_+^n)$ because

$$\begin{aligned}\left|\vec{\lambda}^{\vec{k}+\mathbf{1}} D^{\vec{k}} \Big\langle \Big(\prod_{i=1}^{n} R_i(\lambda_i) \Big) x, x^* \Big\rangle \right| &= \vec{k}! \left| \Big\langle \Big(\prod_{i=1}^{n} \lambda_i^{k_i+1} R_i(\lambda_i)^{k_i+1} \Big) x, x^* \Big\rangle \right| \\ &\le \vec{k}!\, M^{|\vec{k}|+n} ||\vec{x}||\, ||x^*||\end{aligned}$$

for $\forall \vec{\lambda} \in \mathbb{R}_+^n$, and we have

$$\begin{aligned} S(\varphi_{x,x^*}) &= \sum_{|\vec{k}| \le K} (-1)^{|\vec{k}|} \int_{\mathbb{R}_+^n} D^{\vec{k}} \Big\langle \Big(\prod_{i=1}^{n} R_i(\lambda_i) \Big) x, x^* \Big\rangle d\mu_{\vec{k}}(\vec{\lambda}) \\ &= \sum_{|\vec{k}| \le K} \vec{k}! \int_{\mathbb{R}_+^n} \Big\langle \Big(\prod_{i=1}^{n} R_i(\lambda_i)^{k_i+1} \Big) x, x^* \Big\rangle d\mu_{\vec{k}}(\vec{\lambda}) \\ &= \langle \mathcal{R}_R(S)x, x^* \rangle. \end{aligned}$$

□

Lemma 7 $\mathcal{R}_R$ *is well-defined.*

PROOF: Assume there are two representations of the same distribution $S \in \mathcal{D}_{S^*}(\mathbb{R}_+^n)'$ so that the operators assigned to them are not the same. Then the difference of them is a non-trivial representation of the zero-distribution O whose corresponding operator $\mathcal{R}_R(O)$ is not the zero-operator. Therefore there are $x \in X$ and $x^* \in X^*$ with $\langle \mathcal{R}_R(O)x, x^* \rangle \neq 0$. But according to Lemma 6 we have $\langle \mathcal{R}_R(O)x, x^* \rangle = O(\varphi_{x,x^*}) = 0$. Contradiction. □

Note that $\mathcal{R}_R$ is well-defined even if the resolvent families R_i do not commute. Fixing an order for the operators in the expression $\prod_{i=1}^n R_i(\lambda_i)$, this can be useful to prove some non-trivial operator equations of the form $\mathcal{R}_R(S) = 0$ for which the upcoming theorem is not needed.

Theorem 4 *If* $S_1, S_2 \in \mathcal{D}_{S^*}(\mathbb{R}_+^n)'$ *and the resolvent families* R_i, $i = 1, \ldots, n$, *commute, then* $\mathcal{R}_R(S_1 * S_2) = \mathcal{R}_R(S_1)\mathcal{R}_R(S_2)$.

PROOF: Let $x \in X$ and $x^* \in X^*$, and define φ_{x,x^*} as in Lemma 7. Based on the resolvent equation we can generalize Lemma 3 to

$$(\Delta_n \varphi_{x,x^*})(\vec{\lambda}_1, \vec{\lambda}_2) = \Big\langle \prod_{j=1}^{2} \prod_{i=1}^{n} R_i(\lambda_{j,i}) x, x^* \Big\rangle \qquad (\vec{\lambda}_1, \vec{\lambda}_2 \in \mathbb{R}_+^n).$$

Using two representations (8) for S_1 and S_2, equation (20) and Lemma 7, we find that

$$
\begin{aligned}
&\langle \mathcal{R}_R(S_1 * S_2)x, x^*\rangle = (S_1 * S_2)(\varphi_{x,x^*}) \\
&= \sum_{|\vec{k}_1|\le K_1} \sum_{|\vec{k}_2|\le K_2} (-1)^{|\vec{k}_1|+|\vec{k}_2|} \int_{\mathbb{R}^n_+} d\mu^1_{\vec{k}_1}(\vec{\lambda}_1) \int_{\mathbb{R}^n_+} d\mu^2_{\vec{k}_2}(\vec{\lambda}_2) \\
&\qquad\qquad D^{\vec{k}_1}_{\vec{\lambda}_1} D^{\vec{k}_2}_{\vec{\lambda}_2} (\Delta^n \varphi_{x,x^*})(\vec{\lambda}_1, \vec{\lambda}_2) \\
&= \Big\langle \sum_{|\vec{k}_1|\le K_1} \sum_{|\vec{k}_2|\le K_2} \vec{k}_1! \vec{k}_2! \int_{\mathbb{R}^n_+} d\mu^1_{\vec{k}_1}(\vec{\lambda}_1) \int_{\mathbb{R}^n_+} d\mu^2_{\vec{k}_2}(\vec{\lambda}_2) \prod_{j=1}^{2} \prod_{i=1}^{n} R_i(\lambda_{j,i})^{k_{ji}+1} x, x^* \Big\rangle \\
&= \Big\langle \sum_{|\vec{k}_1|\le K_1} \vec{k}_1! \int_{\mathbb{R}^n_+} d\mu^1_{\vec{k}_1}(\vec{\lambda}_1) \prod_{i=1}^{n} R_i(\lambda_{1,i})^{k_{1i}+1} \\
&\qquad\qquad \circ \sum_{|\vec{k}_2|\le K_2} \vec{k}_2! \int_{\mathbb{R}^n_+} d\mu^2_{\vec{k}_2}(\vec{\lambda}_2) \prod_{i=1}^{n} R_i(\lambda_{2,i})^{k_{2i}+1} x, x^* \Big\rangle \\
&= \langle \mathcal{R}_R(S_1)\mathcal{R}_R(S_2)x, x^*\rangle.
\end{aligned}
$$

Since x and x^* were arbitrary, we can conclude $\mathcal{R}_R(S_1 * S_2) = \mathcal{R}_R(S_1)\mathcal{R}_R(S_2)$. □

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